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Nonlinear beating excitations on ladder lattice

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Abstract. The nonlinear model of intramolecular excitations on a ladder lattice integrable by the inverse scattering transform is developed. The model is closely related to the nonlinear Schrödinger model on the same lattice with linear and nonlinear couplings between the chains explicitly taken into account. The pair of auxiliary Lax operators is found and the set of Marchenko-type equations is obtained. The soliton and the reduced soliton solutions of the model are explicitly presented. Even the simplest types of solutions are proved to exhibit both the spatially constricted translational mode typical to the traditional one-chain soliton and the interchain beating mode redistributing the excitations between the chains in a way similar to the linear intramolecular excitations. The possible physical applications of the model are pointed out. The nonlinear model of intramolecular excitations on a multi-leg ladder lattice as well as its continuous counterpart are shown to be integrable too.

1. Introduction

Since the integrability of a nonlinear Schrödinger equation in one spatial dimension was discovered [1] the fundamental role of similar nonlinear models for various physical applications has become generally recognized [2–5]. As a result, a number of applied and purely theoretical problems related to rather different aspects of nonlinear Schrödinger models have been stimulated and successfully solved. Thus, several integrable versions of nonlinear Schrödinger models on discrete chains have been proposed [6–8] and quantized [9, 10]. On the other hand, for the needs of nonlinear optics the integrable multicomponent nonlinear Schrödinger model was developed [11]. Finally, the so-called classical [12] and quantum [13] self-trapping models have also been intensively investigated. Though being nonintegrable, the self-trapping models pretend to describe the nonlinear transport phenomena in a linearly coupled system of chains related to low-dimensional biological objects. However, an integrable model supporting both linear and nonlinear interchain and intrachain couplings of probability amplitudes, to the best of our knowledge, has not been proposed until now. Here we try to fill this gap by presenting an integrable model on a ladder lattice closely reproducing the main features of a discrete nonlinear Schrödinger model with linear and nonlinear couplings between the chains explicitly taken into account.

2. Basic model

In order to make all the necessary definitions in the most natural way, we start our consideration by merely postulating the model of interest:

$$\mathrm{i}\,\mathrm{d}q_\alpha(n)/\mathrm{d}\tau + q_\alpha(n+1) + q_\alpha(n-1) + t q_\beta(n)$$

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$$
\begin{aligned}
&+(q_\alpha(n+1)+q_\alpha(n-1))(q_\alpha(n)r_\alpha(n)+q_\beta(n)r_\beta(n))\\
&+(q_\alpha(n)q_\beta(n-1)-q_\alpha(n-1)q_\beta(n))r_\beta(n)=0 \qquad (1)
\end{aligned}
$$

$$
\begin{aligned}
-\mathrm{i}\mathrm{d}r_\alpha(n)/\mathrm{d}\tau &+ r_\alpha(n+1)+r_\alpha(n-1)+tr_\beta(n)\\
&+(r_\alpha(n+1)+r_\alpha(n-1))(r_\alpha(n)q_\alpha(n)+r_\beta(n)q_\beta(n))\\
&+(r_\alpha(n)r_\beta(n+1)-r_\alpha(n+1)r_\beta(n))q_\beta(n)=0 \qquad (2)
\end{aligned}
$$

and postpone its justification to the next section. Here the indices α and β mark the chain numbers and run over the plus (+) and minus (−) signs in an noncoincident way ($\alpha \neq \beta$), whereas the numerical coordinate n determines the unit cell on the ladder lattice and is supposed to run from minus to plus infinity. Following the terminology of nonlinear transport phenomena we prescribe the quantities τ and t to be time and interchain linear coupling constant, respectively, while $q_\alpha(n)$ and $r_\alpha(n)$ are the excitation amplitudes. Then the nonlinear terms correspond to the intrachain and interchain nonlinear couplings. (Dimensionless units with unity value of the intrachain linear coupling constant is adopted.)

Similar to its one-chain counterpart [6], the two-chain model (1), (2) conserves the quantity $\sum_{m=-\infty}^{\infty} \ln(1+q_+(n)r_+(n)+q_-(n)r_-(n))$ resembling in terms of corrected amplitudes

$$Q_\pm(n)=q_\pm(n)\sqrt{\frac{\ln(1+q_+(n)r_+(n)+q_-(n)r_-(n))}{q_+(n)r_+(n)+q_-(n)r_-(n)}} \qquad (3)$$

$$R_\pm(n)=r_\pm(n)\sqrt{\frac{\ln(1+q_+(n)r_+(n)+q_-(n)r_-(n))}{q_+(n)r_+(n)+q_-(n)r_-(n)}} \qquad (4)$$

the total number of excitations. We have said 'resembling' since, in general, the amplitudes $q_\alpha(n)$ and $r_\alpha(n)$ are not permitted to be linked by the reduction $r_\alpha(n) = q_\alpha^*(n)$, though the one-soliton solution is shown to support the case. We do not presently know whether such a reduction is possible for the multisoliton solutions, but at least for the well separated solitons or any spatially smooth solutions an approximate reduction is always feasible.

3. Auxiliary linear problems

The nonlinear model under study (1), (2) is equivalent to two auxiliary linear problems

$$\boldsymbol{u}(n+1|z)=L(n|z)\boldsymbol{u}(n|z) \qquad (5)$$

$$\mathrm{d}\boldsymbol{u}(n|z)/\mathrm{d}\tau=A(n|z)\boldsymbol{u}(n|z) \qquad (6)$$

on the four-component column vector $\boldsymbol{u}(n|z) \equiv \mathrm{col}[u_1(n|z), u_2(n|z), u_3(n|z), u_4(n|z)]$ with the spectral operator $L(n|z)$ given by the fourth-rank matrix

$$L(n|z)=\begin{pmatrix} z & 0 & \mathrm{i}q_+(n)/\sqrt{2} & \mathrm{i}q_+(n)/\sqrt{2}\\ 0 & z & \mathrm{i}q_-(n)/\sqrt{2} & \mathrm{i}q_-(n)/\sqrt{2}\\ \mathrm{i}r_+(n)/\sqrt{2} & \mathrm{i}r_-(n)/\sqrt{2} & 1/z & 0\\ \mathrm{i}r_+(n)/\sqrt{2} & \mathrm{i}r_-(n)/\sqrt{2} & 0 & 1/z \end{pmatrix} \qquad (7)$$

and the evolution operator $A(n|z)$ following from the compatibility condition

$$[\mathrm{d}\boldsymbol{u}(m|z)/\mathrm{d}\tau]_{m=n+1}=\mathrm{d}\boldsymbol{u}(n+1|z)/\mathrm{d}\tau \qquad (8)$$

and the requirement that the power sequences in the expansions of $A(n|z)$ and $L^2(n|z)$, with respect to z, should coincide. Indeed, the compatibility condition (8) applied to the auxiliary problems (5), (6) gives rise to the Lax equation:

$$\mathrm{d}L(n|z)/\mathrm{d}\tau=A(n+1|z)L(n|z)-L(n|z)A(n|z) \qquad (9)$$

sufficient to both restore the explicit matrix form of evolution operator

$$A_{11}(n|z) = \mathrm{i}z^2 - \mathrm{i}t/2 + \mathrm{i}q_+(n)r_+(n-1) \tag{10}$$

$$A_{12}(n|z) = \mathrm{i}t + \mathrm{i}q_+(n)r_-(n-1) \tag{11}$$

$$A_{21}(n|z) = \mathrm{i}t + \mathrm{i}q_-(n)r_+(n-1) \tag{12}$$

$$A_{22}(n|z) = \mathrm{i}z^2 - \mathrm{i}t/2 + \mathrm{i}q_-(n)r_-(n-1) \tag{13}$$

$$A_{13}(n|z) \equiv A_{14}(n|z) = -zq_+(n)/\sqrt{2} + z^{-1}q_+(n-1)/\sqrt{2} \tag{14}$$

$$A_{23}(n|z) \equiv A_{24}(n|z) = -zq_-(n)/\sqrt{2} + z^{-1}q_-(n-1)/\sqrt{2} \tag{15}$$

$$A_{31}(n|z) \equiv A_{41}(n|z) = z^{-1}r_+(n)/\sqrt{2} - zr_+(n-1)/\sqrt{2} \tag{16}$$

$$A_{32}(n|z) \equiv A_{42}(n|z) = z^{-1}r_-(n)/\sqrt{2} - zr_-(n-1)/\sqrt{2} \tag{17}$$

$$A_{33}(n|z) \equiv A_{44}(n|z) = -\mathrm{i}z^{-2} + \mathrm{i}t/2 - \frac{\mathrm{i}}{2}(r_+(n)q_+(n-1) + r_-(n)q_-(n-1)) \tag{18}$$

$$A_{34}(n|z) \equiv A_{43}(n|z) = -\mathrm{i}t - \frac{\mathrm{i}}{2}(r_+(n)q_+(n-1) + r_-(n)q_-(n-1)) \tag{19}$$

and to isolate the model of interest (1), (2). In the course of the calculations the spectral parameter z was assumed to be time independent, as usual. The explicit indications on the time dependences of other quantities will typically be omitted for the sake of brevity.

4. Inverse scattering scheme. Definitions and preparatory results

Now we are in a position to develop the inverse scattering technique for solving our model (1), (2).

Restricting to the case of the potentials, $q_\pm(n)$ and $r_\pm(n)$, rapidly decreasing at infinity, we define the left $\{\boldsymbol{\varphi}_j(n|z)\}$ and the right $\{\boldsymbol{\psi}_j(n|z)\}$ Jost bases ($j = 1, 2, 3, 4$) as the vector sets satisfying the auxiliary spectral problem (5), (7) and fixed by the asymptotic conditions:

$$\varphi_{ij}(n|z) \sim \delta_{ij}(\delta_{j1} + \delta_{j2})z^n + \delta_{ij}(\delta_{j3} + \delta_{j4})z^{-n} \qquad \text{as} \quad n \to -\infty \tag{20}$$

$$\psi_{ij}(n|z) \sim \delta_{ij}(\delta_{j1} + \delta_{j2})z^n + \delta_{ij}(\delta_{j3} + \delta_{j4})z^{-n} \qquad \text{as} \quad n \to +\infty. \tag{21}$$

Here $\varphi_{ij}(n|z)$ and $\psi_{ij}(n|z)$ are the ith components of vectors $\boldsymbol{\varphi}_j(n|z)$ and $\boldsymbol{\psi}_j(n|z)$, respectively. Then the transition matrix $[a_{jk}(z)]$ is that transforming one basis into another

$$\boldsymbol{\varphi}_k(n|z) = \sum_{j=1}^{4} \boldsymbol{\psi}_j(n|z)a_{jk}(z) \qquad (k = 1, 2, 3, 4). \tag{22}$$

Conversely, supposing that the Jost bases are known, the following relation:

$$a_{ij}(z) = \frac{\operatorname*{W}\limits_{k=1}^{4}\{(1-\delta_{ik})\boldsymbol{\psi}_k(n|z) + \delta_{ik}\boldsymbol{\varphi}_j(n|z)\}}{\operatorname*{W}\limits_{k=1}^{4}\{\boldsymbol{\psi}_k(n|z)\}} \tag{23}$$

for the matrix elements $a_{ij}(z)$ can be easily obtained. Here $\operatorname*{W}\limits_{k=1}^{4}\{\boldsymbol{v}_k(n|z)\}$ stands for the Wronskian of any four solutions $\boldsymbol{v}_1(n|z)$, $\boldsymbol{v}_2(n|z)$, $\boldsymbol{v}_3(n|z)$, $\boldsymbol{v}_4(n|z)$ of the spectral problem (5), (7) and is defined by the identity

$$\operatorname*{W}\limits_{k=1}^{4}\{\boldsymbol{v}_k(n|z)\} \equiv \det[v_{ik}(n|z)] \tag{24}$$

with $v_{ik}(n|z)$ denoting the ith component of vector $\boldsymbol{v}_k(n|z)$.

Further, taking the Wronskian from both parts of the transforming equations (22) we come to the normalizing condition

$$\det[a_{ij}(z)] = \frac{\overset{4}{\underset{k=1}{\mathrm{W}}}\{\varphi_k(n|z)\}}{\overset{4}{\underset{k=1}{\mathrm{W}}}\{\psi_k(n|z)\}} = \prod_{m=-\infty}^{\infty}(1 + q_+(m)r_+(m) + q_-(m)r_-(m)) \qquad (25)$$

where the last step is assisted by the relations:

$$\overset{4}{\underset{k=1}{\mathrm{W}}}\{\varphi_k(n|z)\} = \prod_{m=-\infty}^{n-1}(1 + q_+(m)r_+(m) + q_-(m)r_-(m)) \qquad (26)$$

$$\overset{4}{\underset{k=1}{\mathrm{W}}}\{\psi_k(n|z)\} = \prod_{m=n}^{\infty}(1 + q_+(m)r_+(m) + q_-(m)r_-(m))^{-1} \qquad (27)$$

based upon the combination of the spectral problem (5), (7) and the asymptotic conditions (20), (21).

Finally, it can be shown that two sets of vectors

$$\{\varphi_1(n|z)z^{-n}, \varphi_2(n|z)z^{-n}, \psi_3(n|z)z^n, \psi_4(n|z)z^n\} \qquad (28)$$

and

$$\{\psi_1(n|z)z^{-n}, \psi_2(n|z)z^{-n}, \varphi_3(n|z)z^n, \varphi_4(n|z)z^n\} \qquad (29)$$

are analytic outside $|z| > 1$ and inside $|z| < 1$ the unit circle, respectively, provided the potentials $q_\pm(n)$ and $r_\pm(n)$ abate sufficiently rapidly as $|n| \to \infty$. These properties enable us to seek the vectors of the right Jost basis in the form

$$\psi_j(n|z) = \sum_{l=n}^{\infty} \boldsymbol{K}_j(n|l)[(\delta_{j1} + \delta_{j2})z^l + (\delta_{j3} + \delta_{j4})z^{-l}] \qquad (j = 1, 2, 3, 4). \qquad (30)$$

After being substituted into the spectral equation (5), the expansions (30) yield the relationships between the amplitudes $q_\pm(n)$, $r_\pm(n)$ and the components $K_{ij}(n|m)$ of column vectors $\boldsymbol{K}_j(n|m)$ as follows:

$$q_+(n) = \frac{\mathrm{i}\sqrt{2}K_{13}(n|n+1)}{K_{33}(n|n) + K_{43}(n|n)} \qquad (31)$$

$$q_-(n) = \frac{\mathrm{i}\sqrt{2}K_{23}(n|n+1)}{K_{33}(n|n) + K_{43}(n|n)} \qquad (32)$$

$$r_+(n) = \mathrm{i}\sqrt{2}\frac{K_{31}(n|n+1)K_{22}(n|n) - K_{32}(n|n+1)K_{21}(n|n)}{K_{11}(n|n)K_{22}(n|n) - K_{12}(n|n)K_{21}(n|n)} \qquad (33)$$

$$r_-(n) = \mathrm{i}\sqrt{2}\frac{K_{32}(n|n+1)K_{11}(n|n) - K_{31}(n|n+1)K_{12}(n|n)}{K_{11}(n|n)K_{22}(n|n) - K_{12}(n|n)K_{21}(n|n)}. \qquad (34)$$

Another four relationships are completely equivalent to those presented above and we omit them for the sake of brevity.

At this point it may appear that we could construct the inverse scattering theory by only slightly adjusting the known one-chain results [6]. However, this is by no means the case. In contrast to the one-chain models [1, 6], where the inverse scattering scheme is based on the analytical properties of Jost functions side by side with the analytical properties of diagonal transition coefficients, our situation turns out to be a more complicated one. Indeed, although the analytical properties of Jost vectors are detectable, the analytical properties of transition coefficients *a priori* cannot be revealed and furthermore, prove to be quite unnecessary. Instead, the general logic of the problem leads us to the modified transition matrix $[\alpha_{jk}(z)]$ given by the combinations:

$$\alpha_{jk}(z) = (a_{11}(z)a_{22}(z) - a_{12}(z)a_{21}(z))\delta_{jk} \qquad (\text{at} \quad j = 1, 2;\ k = 1, 2) \tag{35}$$

$$\alpha_{jk}(z) = (a_{jk}(z)a_{22}(z) - a_{j2}(z)a_{2k}(z))\delta_{k1} + (a_{jk}(z)a_{11}(z) - a_{j1}(z)a_{1k}(z))\delta_{k2} \qquad (\text{at} \quad j = 3, 4;\ k = 1, 2) \tag{36}$$

$$\alpha_{jk}(z) = (a_{jk}(z)a_{44}(z) - a_{j4}(z)a_{4k}(z))\delta_{k3} + (a_{jk}(z)a_{33}(z) - a_{j3}(z)a_{3k}(z))\delta_{k4} \qquad (\text{at} \quad j = 1, 2;\ k = 3, 4) \tag{37}$$

$$\alpha_{jk}(z) = (a_{33}(z)a_{44}(z) - a_{34}(z)a_{43}(z))\delta_{jk} \qquad (\text{at} \quad j = 3, 4;\ k = 3, 4). \tag{38}$$

In what follows only the analytical properties of diagonal elements $\alpha_{11}(z) \equiv \alpha_{22}(z)$ and $\alpha_{33}(n) \equiv \alpha_{44}(n)$ of the modified transition matrix are required. Fortunately, they are precisely those elements admitting to thorough treatment. Thus, the expressions

$$\alpha_{11}(z) \equiv \alpha_{22}(z) = \frac{\operatorname*{W}_{k=1}^{4}\{(\delta_{1k} + \delta_{2k})\varphi_k(n|z) + (\delta_{3k} + \delta_{4k})\psi_k(n|z)\}}{\operatorname*{W}_{k=1}^{4}\{\psi_k(n|z)\}} \tag{39}$$

$$\alpha_{33}(z) \equiv \alpha_{44}(z) = \frac{\operatorname*{W}_{k=1}^{4}\{(\delta_{1k} + \delta_{2k})\psi_k(n|z) + (\delta_{3k} + \delta_{4k})\varphi_k(n|z)\}}{\operatorname*{W}_{k=1}^{4}\{\psi_k(n|z)\}} \tag{40}$$

taken at $n \to \infty$ show that $\alpha_{11}(z) \equiv \alpha_{22}(z)$ and $\alpha_{33}(n) \equiv \alpha_{44}(n)$ are analytic outside $|z| > 1$ and inside $|z| < 1$ the unit circle, respectively.

We complete this section by presenting the evolution equations for the elements of the modified transition matrix:

$$\dot{\alpha}_{jk}(z) = -\mathrm{i}(z^2 + z^{-2})\alpha_{jk}(z) - \mathrm{i}t(\alpha_{j1}(z)\delta_{2k} + \alpha_{j2}(z)\delta_{1k}) \qquad (j = 3, 4;\ k = 1, 2) \tag{41}$$

$$\dot{\alpha}_{jk}(z) = \mathrm{i}(z^2 + z^{-2})\alpha_{jk}(z) + \mathrm{i}t(\delta_{j2}\alpha_{1k}(z) + \delta_{j1}\alpha_{2k}(z)) \qquad (j = 1, 2;\ k = 3, 4) \tag{42}$$

$$\dot{\alpha}_{kk}(z) = 0 \qquad (k = 1, 2, 3, 4). \tag{43}$$

Here the dot stands for the derivative with respect to time τ. The evolution equations (41)–(43) have been derived thanks to the standard observation that at every $j = 1, 2, 3, 4$ the combination $\dot{\varphi}_j(n|z) - A(n|z)\varphi_j(n|z)$ satisfies to the spectral problem (5) and consequently, is presentable by some linear superposition of the left Jost vectors.

5. Inverse scattering scheme. Marchenko equations

To proceed with the fundamental aspects of whole inverse scattering scheme it is helpful to rearrange the interbasis link (22) into the from

$$\boldsymbol{S}_k(n|z)\alpha_{kk}(z) = \sum_{j=1}^{4} \psi_j(n|z)\alpha_{jk}(z) \qquad (k = 1, 2, 3, 4) \tag{44}$$

with the scattering vectors $\boldsymbol{S}_k(n|z)$ introduced the following way:

$$\boldsymbol{S}_k(n|z)\alpha_{kk}(z) \equiv \varphi_1(n|z)(\delta_{1k}a_{22}(z) - a_{12}(z)\delta_{2k}) + \varphi_2(n|z)(\delta_{2k}a_{11}(z) - a_{21}(z)\delta_{1k}) + \varphi_3(n|z)(\delta_{3k}a_{44}(z) - a_{34}(z)\delta_{4k}) + \varphi_4(n|z)(\delta_{4k}a_{33}(z) - a_{43}(z)\delta_{3k}). \tag{45}$$

An appropriate analysis of scattering vectors $\boldsymbol{S}_k(n|z)$ similar to that described by Toda [14] yields such limiting formulae

$$(\delta_{1k} + \delta_{2k}) \lim_{|z|\to\infty} \boldsymbol{S}_k(n|z)z^{-n} + (\delta_{3k} + \delta_{4k}) \lim_{|z|\to 0} \boldsymbol{S}_k(n|z)z^n = \boldsymbol{J}_k \qquad (k = 1, 2, 3, 4) \tag{46}$$

and

$$\lim_{|z|\to\infty} a_{11}(z) = \lim_{|z|\to\infty} a_{22}(z) = 1 \tag{47}$$

$$\lim_{|z|\to\infty} a_{12}(z) = \lim_{|z|\to\infty} a_{21}(z) = 0 \tag{48}$$

$$\lim_{|z|\to 0} a_{33}(z) = \lim_{|z|\to 0} a_{44}(z) = 1 \tag{49}$$

$$\lim_{|z|\to 0} a_{34}(z) = \lim_{|z|\to 0} a_{43}(z) = 0 \tag{50}$$

relevant for the future contour integration and reconstruction of diagonal matrix elements $\alpha_{kk}(z)$, respectively. Here $\boldsymbol{J}_k$ refers to the column vector with the ith component equal to δ_{ik}.

Assuming $\alpha_{kk}(z)$ at $|z| = 1$ to be nonzero we operate on the rearranged interbasis relation (44) with

$$\frac{1}{2\pi \mathrm{i}} \oint_{|z|=1} \frac{\mathrm{d}z}{\alpha_{kk}(z)} [(\delta_{1k} + \delta_{2k}) z^{-m-1} + (\delta_{3k} + \delta_{4k}) z^{m-1}] \ldots \tag{51}$$

to find the set of equations

$$\boldsymbol{K}_k(n|m) + \sum_{l=n}^{\infty} \sum_{j=1}^{4} \boldsymbol{K}_j(n|l) F_{jk}(l+m) = \boldsymbol{J}_k \delta_{nm} \qquad (m \geqslant n;\ k = 1, 2, 3, 4) \tag{52}$$

of Marchenko type [15]. Here the matrix elements $F_{jk}(n)$ of the kernel operator are given by the expressions

$$F_{jk}(n) = \frac{1}{2\pi \mathrm{i}} \oint_{|z|=1} \mathrm{d}z\, z^{-n-1} \frac{\alpha_{jk}(z)}{\alpha_{kk}(z)} + \sum_{r=1}^{N_{\text{ext}}} z_{rk}^{-n-1} \frac{\alpha_{jk}(z_{rk})}{\alpha'_{kk}(z_{rk})} \qquad (\text{at} \quad j = 3, 4;\ k = 1, 2) \tag{53}$$

$$F_{jk}(n) = \frac{1}{2\pi \mathrm{i}} \oint_{|z|=1} \mathrm{d}z\, z^{n-1} \frac{\alpha_{jk}(z)}{\alpha_{kk}(z)} - \sum_{r=1}^{N_{\text{int}}} z_{rk}^{n-1} \frac{\alpha_{jk}(z_{rk})}{\alpha'_{kk}(z_{rk})} \qquad (\text{at} \quad j = 1, 2;\ k = 3, 4) \tag{54}$$

$$F_{jk}(n) \equiv 0 \qquad (\text{otherwise}) \tag{55}$$

where z_{rk} stands for the rth root of the equation $\alpha_{kk}(z) = 0$, $\alpha'_{kk}(z_{rk})$ refers to the derivative $[\mathrm{d}\alpha_{kk}(z)/\mathrm{d}z]_{z=z_{rk}}$ while N_{ext} and N_{int} mark the total number of roots of the equations $\alpha_{11}(z) = 0$ and $\alpha_{33}(z) = 0$, respectively. Incidentally, although the equalities (53)–(55) have been found within an unspoken premise of simple roots z_{rk}, the case of multiple ones can evidently be covered by mere limiting passages in final results.

In contrast to the one-chain models the time dependences of scattering data $\alpha_{jk}(z)/\alpha_{kk}(z)$ as well as $\alpha_{jk}(z_{rk})/\alpha'_{kk}(z_{rk})$ and z_{rk} are more sophisticated though still rather simple. Indeed, integrating the evolution equations (41)–(43) and restoring an explicit indication on time $\alpha_{jk}(z) \equiv \alpha_{jk}(z|\tau)$ we obtain

$$\begin{aligned}\alpha_{jk}(z|\tau) = \exp[-\mathrm{i}(z^2 + z^{-2})\tau][(\alpha_{j1}(z|0)\delta_{1k} + \alpha_{j2}(z|0)\delta_{2k}) \cos(t\tau) \\ -\mathrm{i}(\alpha_{j2}(z|0)\delta_{1k} + \alpha_{j1}(z|0)\delta_{2k}) \sin(t\tau)] \qquad (\text{at} \quad j = 3, 4;\ k = 1, 2)\end{aligned} \tag{56}$$

$$\begin{aligned}\alpha_{jk}(z|\tau) = \exp[\mathrm{i}(z^2 + z^{-2})\tau][(\delta_{j1}\alpha_{1k}(z|0) + \delta_{j2}\alpha_{2k}(z|0)) \cos(t\tau) \\ +\mathrm{i}(\delta_{j1}\alpha_{2k}(z|0) + \delta_{j2}\alpha_{1k}(z|0)) \sin(t\tau)] \qquad (\text{at} \quad j = 1, 2;\ k = 3, 4)\end{aligned} \tag{57}$$

$$\alpha_{jk}(z|\tau) = \alpha_{jk}(z|0) \qquad (\text{otherwise}). \tag{58}$$

These formulae are sufficient to extract the time dependences for any spectral data of interest. Thus, it is clearly seen that alongside the purely translational modes evolving as $\exp[\pm\mathrm{i}(z^2+z^{-2})\tau]$ and $\exp[\pm\mathrm{i}(z_{rk}^2+z_{rk}^{-2})\tau]$, the transverse beating mode evolving as $\exp(\pm\mathrm{i}t\tau)$ becomes a distinctive feature of our model.

6. Symmetry of Lax operators and its consequences

It is easily verified that both of the Lax operators obey to the same involuntary transformation

$$PL(n|z)P = L(n|z) \tag{59}$$

$$PA(n|z)P = A(n|z) \tag{60}$$

where $P \equiv P^{-1}$ and the matrix representation of operator P is given by

$$P_{ik} = \delta_{ik} - \delta_{i3}\delta_{3k} - \delta_{i4}\delta_{4k} + \delta_{i3}\delta_{4k} + \delta_{i4}\delta_{3k}. \tag{61}$$

In doing so we must adopt

$$P\varphi_j(n|z) = \varphi_j(n|z) \tag{62}$$

$$P\psi_j(n|z) = \psi_j(n|z) \tag{63}$$

in order to avoid any ambiguities with the asymptotic relations (20), (21). The last two expressions (62) and (63) enable us to both perceive the symmetry of modified transition matrix $[\alpha_{jk}(z)]$ and to establish useful links between the elements of resolving matrix $[K_{ij}(n|m)]$. Namely we have:

$$\alpha_{3k}(z) = \alpha_{4k}(z) \qquad (\text{at} \quad k = 1, 2) \tag{64}$$

$$\alpha_{j3}(z) = \alpha_{j4}(z) \qquad (\text{at} \quad j = 1, 2) \tag{65}$$

$$K_{3j}(n|m) = K_{4j}(n|m) \qquad (\text{at} \quad j = 1, 2) \tag{66}$$

$$K_{i3}(n|m) = K_{i4}(n|m) \qquad (\text{at} \quad i = 1, 2) \tag{67}$$

$$K_{34}(n|m) = K_{43}(n|m) \tag{68}$$

$$K_{33}(n|m) = K_{44}(n|m). \tag{69}$$

7. Soliton solutions

For practical purposes it is worthwhile to separate the equations for $\boldsymbol{K}_1(n|m)$ and $\boldsymbol{K}_2(n|m)$ from those for $\boldsymbol{K}_3(n|m)$ and $\boldsymbol{K}_4(n|m)$ and to reshape the Marchenko equations (52) into the form

$$\boldsymbol{K}_i(n|m) - \sum_{l=n}^{\infty}\sum_{p=n}^{\infty}\sum_{j=1}^{2}\sum_{k=3}^{4} \boldsymbol{K}_j(n|l)F_{jk}(l+p)F_{ki}(p+m) = \boldsymbol{J}_i\delta_{nm} - \sum_{k=3}^{4}\boldsymbol{J}_k F_{ki}(n+m)$$
$$(m \geqslant n;\ i = 1, 2) \tag{70}$$

$$\boldsymbol{K}_i(n|m) - \sum_{l=n}^{\infty}\sum_{p=n}^{\infty}\sum_{j=3}^{4}\sum_{k=1}^{2} \boldsymbol{K}_j(n|l)F_{jk}(l+p)F_{ki}(p+m) = \boldsymbol{J}_i\delta_{nm} - \sum_{k=1}^{2}\boldsymbol{J}_k F_{ki}(n+m)$$
$$(m \geqslant n;\ i = 3, 4). \tag{71}$$

To adapt these equations for the needs of multisolitonic solutions we must equalize the scattering data of the continuous spectrum $\alpha_{jk}(z)/\alpha_{kk}(z)$ to zero, either on and inside $|z| \leqslant 1$ or on and outside $|z| \geqslant 1$ the unit circle depending on what combination of indices $(j = 3, 4;\ k = 1, 2)$ or $(j = 1, 2;\ k = 3, 4)$ are chosen (the so-called unreflectional case). Then, on the one hand the matrix elements $F_{jk}(l+m)$ of the kernel operator become degenerate (see equations (53)–(55)) and on the other the form of diagonal matrix elements $\alpha_{kk}(z)$ can be reconstructed explicitly

$$\alpha_{11}(z) \equiv \alpha_{22}(z) = \prod_{s=1}^{N} \frac{z^2 - \exp(\mu_s + \mathrm{i}p_s)}{z^2 - \exp(-\nu_s + \mathrm{i}q_s)} \tag{72}$$

$$\alpha_{33}(z) \equiv \alpha_{44}(z) = \prod_{s=1}^{N} \frac{z^{-2} - \exp(\nu_s - \mathrm{i}q_s)}{z^{-2} - \exp(-\mu_s - \mathrm{i}p_s)}. \tag{73}$$

Here p_s and q_s are real constants, whereas μ_s and ν_s are positive real constants. Except for the restrictions imposed by the assumed simplicity of roots z_{rk} the constants p_s, q_s, μ_s, ν_s are supposed to be arbitrary in all other respects. Finally, N marks an arbitrary but fixed positive integer being the number of solitons in some particular multisoliton solution. Evidently $N_{\text{ext}} = N_{\text{int}} = 2N$.

Having been presented for the unreflectional case the expressions (72) and (73) are consistent with the analyticity conditions ($\alpha_{11}(z)$ is analytical at $|z| > 1$ and $\alpha_{33}(z)$ is analytical at $|z| < 1$), the limiting conditions ($\lim_{|z|\to\infty} \alpha_{11}(z) = 1$ and $\lim_{|z|\to 0} \alpha_{33}(z) = 1$) as well as with the normalizing condition (25) and the parity conditions $\alpha_{kk}(-z) = \alpha_{kk}(z)$. Though not mentioned earlier, the conditions $\alpha_{kk}(-z) = \alpha_{kk}(z)$ can easily be proved, at least for rapidly abating potentials $q_\pm(n)$ and $r_\pm(n)$ close to those on the compact support. We observe, in passing, that all other nonzero elements of the modified transition matrix happen to be odd functions of the spectral parameter $\alpha_{jk}(-z) = -\alpha_{jk}(z)$ ($j = 3, 4$; $k = 1, 2$ and $j = 1, 2$; $k = 3, 4$).

Manipulating the Marchenko equations (70), (71) in a way standard to integral equations with degenerate kernel [16] and exploring all parity conditions of the modified transition matrix just referred to, we find:

$$\begin{aligned} \boldsymbol{K}_i(n|m) = {} & \sum_{s'=1}^{N} \sum_{j=3}^{4} \boldsymbol{X}_j^{s'q}(n) \sin^2\left(\frac{\pi m}{2}\right) \sum_{s''=1}^{N} \sum_{k=1}^{2} \exp(-\eta_{s''r} m) C_{s'qs''r}(n) b_{jk}^{s'q} b_{ki}^{s''r} \\ & + \sum_{s'=1}^{N} \sum_{j=3}^{4} \boldsymbol{Y}_j^{s'q}(n) \cos^2\left(\frac{\pi m}{2}\right) \\ & \times \sum_{s''=1}^{N} \sum_{k=1}^{2} \exp(-\eta_{s''r} m) S_{s'qs''r}(n) b_{jk}^{s'q} b_{ki}^{s''r} \\ & + \boldsymbol{J}_i \delta_{nm} - \left[\sin^2\left(\frac{\pi n}{2}\right) \cos^2\left(\frac{\pi m}{2}\right) + \cos^2\left(\frac{\pi n}{2}\right) \sin^2\left(\frac{\pi m}{2}\right)\right] \\ & \times \sum_{s'=1}^{N} \exp[-\eta_{s'r}(n+m)] \sum_{k=1}^{2} \boldsymbol{J}_k b_{ki}^{s'r} \qquad (m \geqslant n;\ i = 3, 4) \end{aligned} \tag{74}$$

$$\begin{aligned} \boldsymbol{K}_i(n|m) = {} & \sum_{s'=1}^{N} \sum_{j=1}^{2} \boldsymbol{X}_j^{s'r}(n) \sin^2\left(\frac{\pi m}{2}\right) \sum_{s''=1}^{N} \sum_{k=3}^{4} \exp(-\eta_{s''q} m) C_{s'rs''q}(n) b_{jk}^{s'r} b_{ki}^{s''q} \\ & + \sum_{s'=1}^{N} \sum_{j=1}^{2} \boldsymbol{Y}_j^{s'r}(n) \cos^2\left(\frac{\pi m}{2}\right) \\ & \times \sum_{s''=1}^{N} \sum_{k=3}^{4} \exp(-\eta_{s''q} m) S_{s'rs''q}(n) b_{jk}^{s'r} b_{ki}^{s''q} \\ & + \boldsymbol{J}_i \delta_{nm} - \left[\sin^2\left(\frac{\pi n}{2}\right) \cos^2\left(\frac{\pi m}{2}\right) + \cos^2\left(\frac{\pi n}{2}\right) \sin^2\left(\frac{\pi m}{2}\right)\right] \\ & \times \sum_{s'=1}^{N} \exp[-\eta_{s'q}(n+m)] \sum_{k=3}^{4} \boldsymbol{J}_k b_{ki}^{s'q} \qquad (m \geqslant n;\ i = 1, 2) \end{aligned} \tag{75}$$

where the four-component column vectors $\boldsymbol{X}_i^{sq}(n)$, $\boldsymbol{Y}_i^{sq}(n)$ with $i = 3, 4$ and $\boldsymbol{X}_i^{sr}(n)$, $\boldsymbol{Y}_i^{sr}(n)$

with $i = 1, 2$ are determined from the following four sets of linear algebraic equations:

$$\boldsymbol{X}_i^{sq}(n) - \sum_{s'=1}^{N}\sum_{j=3}^{4}\boldsymbol{X}_j^{s'q}(n)\sum_{s''=1}^{N}\sum_{k=1}^{2}C_{s'qs''r}(n)S_{s''rsq}(n)b_{jk}^{s'q}b_{ki}^{s''r}$$
$$= \sin^2\left(\frac{\pi n}{2}\right)\exp(-\eta_{sq}n)\boldsymbol{J}_i - \cos^2\left(\frac{\pi n}{2}\right)$$
$$\times\sum_{s'=1}^{N}\exp(-\eta_{s'r}n)S_{s'rsq}(n)\sum_{k=1}^{2}\boldsymbol{J}_k b_{ki}^{s'r} \qquad (i = 3, 4) \tag{76}$$

$$\boldsymbol{Y}_i^{sq}(n) - \sum_{s'=1}^{N}\sum_{j=3}^{4}\boldsymbol{Y}_j^{s'q}(n)\sum_{s''=1}^{N}\sum_{k=1}^{2}S_{s'qs''r}(n)C_{s''rsq}(n)b_{jk}^{s'q}b_{ki}^{s''r}$$
$$= \cos^2\left(\frac{\pi n}{2}\right)\exp(-\eta_{sq}n)\boldsymbol{J}_i - \sin^2\left(\frac{\pi n}{2}\right)$$
$$\times\sum_{s'=1}^{N}\exp(-\eta_{s'r}n)C_{s'rsq}(n)\sum_{k=1}^{2}\boldsymbol{J}_k b_{ki}^{s'r} \qquad (i = 3, 4) \tag{77}$$

$$\boldsymbol{X}_i^{sr}(n) - \sum_{s'=1}^{N}\sum_{j=1}^{2}\boldsymbol{X}_j^{s'r}(n)\sum_{s''=1}^{N}\sum_{k=3}^{4}C_{s'rs''q}(n)S_{s''qsr}(n)b_{jk}^{s'r}b_{ki}^{s''q}$$
$$= \sin^2\left(\frac{\pi n}{2}\right)\exp(-\eta_{sr}n)\boldsymbol{J}_i - \cos^2\left(\frac{\pi n}{2}\right)$$
$$\times\sum_{s'=1}^{N}\exp(-\eta_{s'q}n)S_{s'qsr}(n)\sum_{k=3}^{4}\boldsymbol{J}_k b_{ki}^{s'q} \qquad (i = 1, 2) \tag{78}$$

$$\boldsymbol{Y}_i^{sr}(n) - \sum_{s'=1}^{N}\sum_{j=1}^{2}\boldsymbol{Y}_j^{s'r}(n)\sum_{s''=1}^{N}\sum_{k=3}^{4}S_{s'rs''q}(n)C_{s''qsr}(n)b_{jk}^{s'r}b_{ki}^{s''q}$$
$$= \cos^2\left(\frac{\pi n}{2}\right)\exp(-\eta_{sr}n)\boldsymbol{J}_i - \sin^2\left(\frac{\pi n}{2}\right)$$
$$\times\sum_{s'=1}^{N}\exp(-\eta_{s'q}n)C_{s'qsr}(n)\sum_{k=3}^{4}\boldsymbol{J}_k b_{ki}^{s'q} \qquad (i = 1, 2) \tag{79}$$

respectively. Here the notations

$$\eta_{sq} = \tfrac{1}{2}(\mu_s + \mathrm{i}p_s) \tag{80}$$

$$\eta_{sr} = \tfrac{1}{2}(\nu_s - \mathrm{i}q_s) \tag{81}$$

$$C_{s'qs''r}(n) \equiv C_{s''rs'q}(n) = 2\left[\exp(\eta_{s'q} + \eta_{s''r})\cos^2\left(\frac{\pi n}{2}\right) + \sin^2\left(\frac{\pi n}{2}\right)\right]\frac{\exp[-(\eta_{s'q} + \eta_{s''r})n]}{\mathrm{sh}(\eta_{s'q} + \eta_{s''r})} \tag{82}$$

$$S_{s'qs''r}(n) \equiv S_{s''rs'q}(n) = 2\left[\exp(\eta_{s'q} + \eta_{s''r})\sin^2\left(\frac{\pi n}{2}\right) + \cos^2\left(\frac{\pi n}{2}\right)\right]\frac{\exp[-(\eta_{s'q} + \eta_{s''r})n]}{\mathrm{sh}(\eta_{s'q} + \eta_{s''r})} \tag{83}$$

$$b_{jk}^{sq} = 2\frac{\alpha_{jk}(\exp(\eta_{sq}))}{\alpha'_{kk}(\exp(\eta_{sq}))}\exp(-\eta_{sq}) \qquad (j = 3, 4;\ k = 1, 2) \tag{84}$$

$$b_{jk}^{sr} = -2\frac{\alpha_{jk}(\exp(-\eta_{sr}))}{\alpha'_{kk}(\exp(-\eta_{sr}))}\exp(\eta_{sr}) \qquad (j = 1, 2;\ k = 3, 4) \tag{85}$$

are adopted.

In principle, the formulae (74)–(79) supplemented by relations (31)–(34) between $K_{ij}(n|m)$ and $q_\pm(n)$, $r_\pm(n)$ unravel the problem of any multisolitonic solution of our nonlinear model (1), (2).

For example, the amplitudes of one soliton solution (i.e. soliton solution with $N = 1$) are found to be

$$\begin{aligned}\begin{Bmatrix} q_+(n) \\ q_-(n) \end{Bmatrix} &= \mathrm{sh}\left(\frac{\mu+\nu+\mathrm{i}p-\mathrm{i}q}{2}\right) \\ &\quad\cdot \mathrm{sech}\left[\frac{\mu+\nu+\mathrm{i}p-\mathrm{i}q}{2}(n-x-\mathrm{i}y)+\mathrm{i}\tau\,\mathrm{ch}(\mu+\mathrm{i}p)-\mathrm{i}\tau\,\mathrm{ch}(\nu-\mathrm{i}q)\right] \\ &\quad\cdot \exp\left[\frac{\mu-\nu+\mathrm{i}p+\mathrm{i}q}{2}n+\mathrm{i}\tau\,\mathrm{ch}(\mu+\mathrm{i}p)+\mathrm{i}\tau\,\mathrm{ch}(\nu-\mathrm{i}q)\right] \\ &\quad\cdot \left[\mathrm{e}^{\gamma_++\mathrm{i}\theta_+}\cos(\theta+\mathrm{i}\gamma)\begin{Bmatrix} \cos(t\tau) \\ \mathrm{i}\sin(t\tau) \end{Bmatrix}+\mathrm{e}^{-\gamma_--\mathrm{i}\theta_-}\sin(\theta+\mathrm{i}\gamma)\begin{Bmatrix} \mathrm{i}\sin(t\tau) \\ \cos(t\tau) \end{Bmatrix}\right]\end{aligned} \tag{86}$$

$$\begin{aligned}\begin{Bmatrix} r_+(n) \\ r_-(n) \end{Bmatrix} &= \mathrm{sh}\left(\frac{\nu+\mu-\mathrm{i}q+\mathrm{i}p}{2}\right) \\ &\quad\cdot \mathrm{sech}\left[\frac{\nu+\mu-\mathrm{i}q+\mathrm{i}p}{2}(n-x-\mathrm{i}y)-\mathrm{i}\tau\,\mathrm{ch}(\nu-\mathrm{i}q)+\mathrm{i}\tau\,\mathrm{ch}(\mu+\mathrm{i}p)\right] \\ &\quad\cdot \exp\left[\frac{\nu-\mu-\mathrm{i}q-\mathrm{i}p}{2}n-\mathrm{i}\tau\,\mathrm{ch}(\nu-\mathrm{i}q)-\mathrm{i}\tau\,\mathrm{ch}(\mu+\mathrm{i}p)\right] \\ &\quad\cdot \left[\mathrm{e}^{-\gamma_+-\mathrm{i}\theta_+}\cos(\theta+\mathrm{i}\gamma)\begin{Bmatrix} \cos(t\tau) \\ -\mathrm{i}\sin(t\tau) \end{Bmatrix}+\mathrm{e}^{\gamma_-+\mathrm{i}\theta_-}\sin(\theta+\mathrm{i}\gamma)\begin{Bmatrix} -\mathrm{i}\sin(t\tau) \\ \cos(t\tau) \end{Bmatrix}\right].\end{aligned} \tag{87}$$

Here μ, ν, p, q, x, y as well as $\gamma_\pm$, $\theta_\pm$ and γ, θ are real integration parameters linked to the scattering data of discrete spectrum $\alpha_{jk}\left(\exp(\eta_{sq})\right)/\alpha'_{kk}\left(\exp(\eta_{sq})\right)$, $\alpha_{jk}\left(\exp(-\eta_{sr})\right)/\alpha'_{kk}\left(\exp(-\eta_{sr})\right)$ and $\exp(\eta_{sq})$, $\exp(-\eta_{sr})$ at $\tau = 0$ by some one-to-one relations.

It is clearly seen that the one-soliton amplitudes (86), (87) cancel the nonlinear terms $(q_\alpha(n)q_\beta(n-1)-q_\alpha(n-1)q_\beta(n))r_\beta(n)$ and $(r_\alpha(n)r_\beta(n+1)-r_\alpha(n+1)r_\beta(n))q_\beta(n)$ identically, and actually convert the initial model (1), (2) into that where the reduction $r_\alpha(n) = q^*_\alpha(n)$ is justified (the last step $r_\alpha(n) = q^*_\alpha(n)$ requires the coupling parameter t to be real). Evidently such a reduction simplifies the original one-soliton solution (86), (87) and yields

$$\begin{aligned}\begin{Bmatrix} q_+(n) \\ q_-(n) \end{Bmatrix} &= \mathrm{sh}\,\mu\cdot\mathrm{sech}[\mu(n-x)-2\tau\,\mathrm{sh}\,\mu\sin p]\exp(\mathrm{i}pn-2\mathrm{i}\tau\,\mathrm{ch}\,\mu\cos p) \\ &\quad\cdot\left[\begin{Bmatrix} \cos(t\tau) \\ \mathrm{i}\sin(t\tau) \end{Bmatrix}\mathrm{e}^{\mathrm{i}\theta_+}\cos\theta+\begin{Bmatrix} \mathrm{i}\sin(t\tau) \\ \cos(t\tau) \end{Bmatrix}\mathrm{e}^{-\mathrm{i}\theta_-}\sin\theta\right]\end{aligned} \tag{88}$$

$$r_\pm(n) = q^*_\pm(n). \tag{89}$$

However, despite the reduction the parametrization of this solution, $\{\mu, \theta, p, x, \theta_+, \theta_-\}$ remains much richer than that related to one-chain models [1, 6, 8].

The expressions (86), (87) and (88), (89) allow us to conclude that the two-chain model (1), (2) correctly describes both the spatially constricted translational mode typical to the traditional soliton and the interchain beating mode redistributing the excitations between the chains. In the case of the reduced one-soliton solution (88), (89) the beating amplitude is equal to $\sqrt{\cos^2 2\theta+\sin^2 2\theta\sin^2(\theta_++\theta_-)}$ and can be varied from zero to unity. Conversely, the beating frequency t/π has a fundamental physical origin and is determined exclusively by the interchain coupling constant regardless of any particular solution with real t.

8. Concluding remarks

In this paper we have shown how to include the main physical features of the nonlinear Schrödinger model on a ladder lattice in the inverse scattering scheme and to obtain the corresponding integrable model. Similar to its nonintegrable prototypes our model is expected to be useful for the analysis of nonlinear transport phenomena in low-dimensional biological and condensed matter systems as well as for the needs of nonlinear optics. Throughout the paper we have covered practically all the basic aspects of the model except for the problem of conservation laws. Being infinite in number, the sequence of these laws follows from the expansions $\ln \alpha_{11}(z)$ and $\ln \alpha_{33}(z)$ with respect to inverse and direct powers of spectral parameter, respectively. We are planning to treat the conserving quantities in future investigations in as much as at least the first few of them should be responsible for the Hamiltonian structure of our model.

At this point in time the first version of our paper is complete. However, very recently (more precisely one week before and three days after submission) two papers [17, 18] dealing with the integrable models

$$\mathrm{i}\dot{q}_\alpha(n) + (q_\alpha(n+1) + q_\alpha(n-1))\left(1 + \sum_{\beta=1}^{M} q_\beta(n) q_\beta^*(n)\right) = 0 \qquad \alpha = 1, 2, 3, \ldots, M \tag{90}$$

and

$$\mathrm{i}\dot{q}_\alpha(n) + q_\alpha(n+1) + q_\alpha(n-1) + \sum_{\beta=1}^{M} (q_\alpha(n+1) r_\beta(n) q_\beta(n) + q_\alpha(n) r_\beta(n) q_\beta(n-1)) = 0 \tag{91}$$

$$-\mathrm{i}\dot{r}_\alpha(n) + r_\alpha(n+1) + r_\alpha(n-1) + \sum_{\beta=1}^{M} (r_\beta(n+1) q_\beta(n) r_\alpha(n) + r_\beta(n) q_\beta(n) r_\alpha(n-1)) = 0 \tag{92}$$

$$\alpha = 1, 2, 3, \ldots, M$$

have appeared in the literature. Although being multicomponent ones, these models do not contain the terms responsible for the interchain linear couplings and at first sight are merely successfully handled discretizations of the well known multicomponent nonlinear Schrödinger equation [4, 11]

$$\mathrm{i}\partial \mathcal{Q}_\alpha/\partial\tau + \partial^2 \mathcal{Q}_\alpha/\partial x^2 + 2\sum_{\beta=1}^{M} \mathcal{Q}_\beta \mathcal{R}_\beta \mathcal{Q}_\alpha = 0 \qquad \alpha = 1, 2, 3, \ldots, M \tag{93}$$

with $\mathcal{R}_\beta = \pm \mathcal{Q}_\beta^*$.

Nevertheless, the model (90) was shown to exhibit a novel property referred to as periodic oscillations of the soliton shape [17] or, briefly, soliton-shape oscillations. The question arises whether this purely nonlinear effect is similar to the effect of interchain beating supported, as we know, exclusively by the interchain linear coupling. To reply, we observe that the one-soliton solution $q_\alpha(n)_{\mathrm{sol}}$ ($\alpha = 1, 2, 3, \ldots, M$) of (90) suggested in [17] is nothing but the set of an appropriately chosen ($\sum_{\beta=1}^{M} a_\beta b_\beta = 0$) linear combinations:

$$q_\alpha(n)_{\mathrm{sol}} = \frac{a_\alpha q(n)_{\mathrm{sol}} + b_\alpha^*(-1)^n q^*(n)_{\mathrm{sol}}}{\sqrt{\sum_{\beta=1}^{M}(|a_\beta|^2 + |b_\beta|^2)}} \tag{94}$$

of one-soliton

$$q(n)_{\rm sol} = \operatorname{sh}\mu\,\operatorname{sech}[\mu(n-x) - 2\tau\operatorname{sh}\mu\sin p]\exp[\mathrm{i}pn - 2\mathrm{i}\tau\operatorname{ch}\mu\cos p] \tag{95}$$

and staggered one-soliton $(-1)^n q^*(n)_{\rm sol}$ solutions of the purely one-chain model

$$\mathrm{i}\dot{q}(n) + (q(n+1) + q(n-1))(1 + q^*(n)q(n)) = 0. \tag{96}$$

As a result, the amplitudes of the soliton-shape oscillations $2(-1)^n|a_\alpha b_\alpha|$ also become staggered, mainly giving rise to the redistribution of soliton density within each particular chain rather than to the interchain beating. Of course, the soliton-shape oscillations are possible only in multi-component nonlinear models favourable to the staggering-type solutions, particularly in discrete ones with the symmetry $r_\alpha(n) = \pm q^*_\alpha(n)$ and a very special type of nonlinearity (e.g. in (90)). Summarizing, we see that the effect of interchain beating actually has nothing to do with that of soliton-shape oscillations.

Further, under the mutual constraints $t = 0$ and $M = 2$ the models (1), (2) and (91), (92) cast into each other. But despite the expected coincidence, our one-soliton solution (86), (87) with $t = 0$ turned out to be somewhat richer than that obtained in [18] with $M = 2$ by the Hirota method.

It is interesting to note that alongside with the thoroughly treated multicomponent model (90), [17] has presented a more general integrable model referred to as the semi-discrete matrix nonlinear Schrödinger equations. But despite their affected generality these equations can never be reduced to our model (1), (2) in view of the rather tough restrictions imposed on the submatrices involved. In particular, the submatrices F_1, F_2 and Q_n, R_n from [17] are assumed to be nonsingular (this fact is actually declared by the equations (2.5), (2.14), (2.16) and (2.19) from [17]). In contrast, we relinquished similar assumptions from the very beginning, after revealing that they simply kill the whole idea of interchain linear coupling. In the meantime, the general model naturally casting into (1), (2) reads as follows:

$$\begin{aligned}
&\mathrm{i}\dot{q}_\alpha(n) + q_\alpha(n+1) + q_\alpha(n-1) + \sum_{\beta=1}^{M}\Omega_{\alpha\beta}q_\beta(n)\\
&\qquad + \sum_{\beta=1}^{M}(q_\alpha(n+1)r_\beta(n)q_\beta(n) + q_\alpha(n)r_\beta(n)q_\beta(n-1)) = 0 \qquad (97)\\
&-\mathrm{i}\dot{r}_\alpha(n) + r_\alpha(n+1) + r_\alpha(n-1) + \sum_{\beta=1}^{M}r_\beta(n)\Omega_{\beta\alpha}\\
&\qquad + \sum_{\beta=1}^{M}(r_\beta(n+1)q_\beta(n)r_\alpha(n) + r_\beta(n)q_\beta(n)r_\alpha(n-1)) = 0 \qquad (98)\\
&\qquad \alpha = 1, 2, 3, \ldots, M
\end{aligned}$$

where the submatrix $[\Omega_{\alpha\beta}]$ is supposed to be an arbitrary $M \times M$ matrix independent of the coordinate n. The model (97), (98) proves to be integrable in as much as it permits the Lax-type representation (9) with the auxiliary linear operators $L(n|z)$ and $A(n|z)$ given by the block-matrices

$$L(n|z) = \begin{pmatrix} zI & F(n)E \\ EG(n) & z^{-1}I \end{pmatrix} \tag{99}$$

and

$$A(n|z) = \begin{pmatrix} \overset{--}{A}(n|z) & \overset{-+}{A}(n|z) \\ \overset{+-}{A}(n|z) & \overset{++}{A}(n|z) \end{pmatrix} \tag{100}$$

with

$$\overset{--}{A}(n|z) = \mathrm{i}z^2 I - \mathrm{i}F(n)EEG(n-1) + \mathrm{i}\Omega - \mathrm{i}(\sigma + \chi M)I \tag{101}$$

$$\overset{-+}{A}(n|z) = \mathrm{i}zF(n)E - \mathrm{i}z^{-1}F(n-1)E \tag{102}$$

$$\overset{+-}{A}(n|z) = \mathrm{i}zEG(n-1) - \mathrm{i}z^{-1}EG(n) \tag{103}$$

$$\overset{++}{A}(n|z) = -\mathrm{i}z^{-2}I + \mathrm{i}EG(n)F(n-1)E - \mathrm{i}\sigma I - \mathrm{i}\chi E. \tag{104}$$

Here the quantities I, E, Ω and $F(n)$, $G(n)$ stand for $M \times M$ submatrices defined by

$$I \equiv [I_{\alpha\beta}] = [\delta_{\alpha\beta}] \tag{105}$$

$$E \equiv [E_{\alpha\beta}] = [1] \tag{106}$$

$$\Omega \equiv [\Omega_{\alpha\beta}] \tag{107}$$

$$F(n) \equiv [F_{\alpha\beta}(n)] = [\mathrm{i}q_\alpha(n)\delta_{\alpha\beta}]/\sqrt{M} \tag{108}$$

$$G(n) \equiv [G_{\alpha\beta}(n)] = [\mathrm{i}\delta_{\alpha\beta}r_\beta(n)]/\sqrt{M} \tag{109}$$

while σ and χ mark arbitrary c-numbers independent of the coordinate n. We call the model (97), (98) the nonlinear model of intramolecular excitations on a multi-leg ladder lattice. It is easily checked that at $M \geqslant 2$ the submatrices $F(n)E$ and $EG(n)$ involved in the spectral operator (99) are essentially singular ones.

Incidentally, the continuous equivalent of multi-leg ladder model (97), (98)

$$\mathrm{i}\partial\mathcal{Q}_\alpha/\partial\tau + \partial^2\mathcal{Q}_\alpha/\partial x^2 + \sum_{\beta=1}^{M}\Omega_{\alpha\beta}\mathcal{Q}_\beta + 2\sum_{\beta=1}^{M}\mathcal{Q}_\alpha\mathcal{R}_\beta\mathcal{Q}_\beta = 0 \tag{110}$$

$$-\mathrm{i}\partial\mathcal{R}_\alpha/\partial\tau + \partial^2\mathcal{R}_\alpha/\partial x^2 + \sum_{\beta=1}^{M}\mathcal{R}_\beta\Omega_{\beta\alpha} + 2\sum_{\beta=1}^{M}\mathcal{R}_\beta\mathcal{Q}_\beta\mathcal{R}_\alpha = 0 \tag{111}$$

$$\alpha = 1, 2, 3, \ldots, M$$

also happens to be integrable. But in contrast to its discrete counterpart (97), (98) it enables one to obtain the main physically important conserved quantities:

$$\mathcal{N} = \int_{-\infty}^{\infty}\mathrm{d}x\sum_{\alpha=1}^{M}\mathcal{R}_\alpha\mathcal{Q}_\alpha \tag{112}$$

$$\mathcal{P} = -\frac{\mathrm{i}}{2}\int_{-\infty}^{\infty}\mathrm{d}x\sum_{\alpha=1}^{M}\left[\mathcal{R}_\alpha\frac{\partial\mathcal{Q}_\alpha}{\partial x} - \frac{\partial\mathcal{R}_\alpha}{\partial x}\mathcal{Q}_\alpha\right] \tag{113}$$

$$\mathcal{H} = \int_{-\infty}^{\infty}\mathrm{d}x\left[\sum_{\alpha=1}^{M}\frac{\partial\mathcal{R}_\alpha}{\partial x}\frac{\partial\mathcal{Q}_\alpha}{\partial x} - \sum_{\alpha=1}^{M}\sum_{\beta=1}^{M}\mathcal{R}_\alpha\Omega_{\alpha\beta}\mathcal{Q}_\beta - \left(\sum_{\gamma=1}^{M}\mathcal{R}_\gamma\mathcal{Q}_\gamma\right)^2\right] \tag{114}$$

in a routine way known from standard quantum mechanics. It is clearly seen that the quantity $\mathcal{H}$ explicitly contains the linear coupling coefficients $\Omega_{\alpha\beta}$. Nothing similar can be obtained standing on the general positions of Wadati's team [17, 19], which once again confirms the principal distinctions between our models (1), (2); (97), (98); (110), (111) and the semi-discrete matrix nonlinear Schrödinger equations [17]. As a matter of fact, the set of conservation laws obtained for the semi-discrete nonlinear Schrödinger equations [17] becomes invalid as applied to our models (1), (2) and (97), (98) and we ought to derive the new one under absolutely different assumptions.

In conclusion, we prove the integrability of continuous model (110), (111) starting with the auxiliary linear operators $\mathcal{L}(x|\lambda)$ and $\mathcal{A}(x|\lambda)$ given by

$$\mathcal{L}(x|\lambda) = \begin{pmatrix} \mathrm{i}\lambda I & \mathcal{F}(x)E \\ E\mathcal{G}(x) & -\mathrm{i}\lambda I \end{pmatrix} \tag{115}$$

$$\mathcal{A}(x|\lambda) = \begin{pmatrix} \overset{--}{\mathcal{A}}(x|\lambda) & \overset{-+}{\mathcal{A}}(x|\lambda) \\ \overset{+-}{\mathcal{A}}(x|\lambda) & \overset{++}{\mathcal{A}}(x|\lambda) \end{pmatrix}. \tag{116}$$

Here

$$\overset{--}{\mathcal{A}}(x|\lambda) = -2\mathrm{i}\lambda^2 I - \mathrm{i}\mathcal{F}(x)EE\mathcal{G}(x) + \mathrm{i}\Omega - \mathrm{i}(\sigma + \chi M)I \tag{117}$$

$$\overset{-+}{\mathcal{A}}(x|\lambda) = -2\lambda\mathcal{F}(x)E + \mathrm{i}\partial\mathcal{F}(x)/\partial x E \tag{118}$$

$$\overset{+-}{\mathcal{A}}(x|\lambda) = -2\lambda E\mathcal{G}(x) - \mathrm{i}E\partial\mathcal{G}(x)/\partial x \tag{119}$$

$$\overset{++}{\mathcal{A}}(x|\lambda) = 2\mathrm{i}\lambda^2 I + \mathrm{i}E\mathcal{G}(x)\mathcal{F}(x)E - \mathrm{i}\sigma I - \mathrm{i}\chi E \tag{120}$$

where the quantities I, E, Ω and $\mathcal{F}(x)$, $\mathcal{G}(x)$ are assumed to be $M \times M$ submatrices defined by the expressions (105)–(107) and

$$\mathcal{F}(x) \equiv [\mathcal{F}_{\alpha\beta}(x)] = [\mathrm{i}\mathcal{Q}_\alpha(x)\delta_{\alpha\beta}]/\sqrt{M} \tag{121}$$

$$\mathcal{G}(x) \equiv [\mathcal{G}_{\alpha\beta}(x)] = [\mathrm{i}\delta_{\alpha\beta}\mathcal{R}_\beta(x)]/\sqrt{M} \tag{122}$$

respectively, while λ stands for the time-independent spectral parameter. The coordinate independent c-numbers, σ and χ, can always be equalized to zero without loss of generality. The model (110), (111) follows from the zero-curvature condition

$$\partial\mathcal{L}(x|\lambda)/\partial\tau = \partial\mathcal{A}(x|\lambda)/\partial x + \mathcal{A}(x|\lambda)\mathcal{L}(x|\lambda) - \mathcal{L}(x|\lambda)\mathcal{A}(x|\lambda) \tag{123}$$

thereby indicating its integrability.

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